ON THE STABILITY OF TRANSVERSE FLOW OF FLUID BETWEEN PERMEABLE BOUNDARIES

(OB USTOICHIVOSTI POPERECHNOGO TECHENIIA ZHIDKOSTI MEZHDU PRONITSAEMYMI GRANITSAMI)

PMM Vol. 31, No. 1, 1967, pp. 116-119

G. Z. GERSHUNI, E. M. ZHUKHOVITSKII and D. L. SHVARTSBLAT
(Perm')

(Received October 1, 1966)

It is a well known fact that the investigation of stability of steady flows of viscous fluid presents formidable mathematical difficulties. Exact solution of the problem dealing with the spectrum of small normal perturbations is known only in some isolated cases. Below we consider an example of a simple steady motion for which the problem of the perturbation spectrum is solved exactly.

Let us consider a plane infinite layer of viscous incompressible fluid, bounded by the planes $z = \pm h$. We shall assume that the boundaries of the layer are permeable and that through the plane $z = -h$ a homogeneous flow of fluid takes place with the velocity v_0 in the inward direction, while across the plane $z = h$, we have the same flow in the outward direction. Thus we have, in the fluid layer, a steady transverse motion with homogeneous velocity

$$v_x = 0, \qquad v_y = 0, \qquad v_z = v_0 \tag{1}$$

(we have chosen the origin to be in the center of the layer with x- and y-axes parallel to the boundaries

Equations of small perturbations of the steady motion (1) are obtained in the usual manner from the Navier-Stokes and continuity equations. Eliminating the pressure and x- and y-components of the perturbation velocity, we obtain the equation for the transverse component of the perturbation velocity $v_z = v_z(x, y, z, t)$

$$\frac{\partial}{\partial t}\Delta v_z + R\frac{\partial}{\partial z}\Delta v_z = \Delta^2 v_z, \qquad R = \frac{v_0 h}{\nu} \tag{2}$$

This is written in the dimensionless form and h, h^2/ν and v_0 are taken as the units of distance, time and velocity, respectively. A dimensionless parameter entering (2) is the Reynolds' number R.

Let us consider normal perturbations of the type

$$v_z = v(z) \ \exp\left[-\lambda t + i(k_1 x + k_2 y)\right] \tag{3}$$

periodic in the plane of the layer.

Here λ is the complex perturbation decrement, while k_1 and k_2 are wave numbers in the x- and y-direction. From (2) we obtain the equation for the amplitude of perturbations $v(z)$

$$-\lambda (v'' - k^2 v) + R (v''' - k^2 v') = v^{IV} - 2k^2 v'' + k^4 v, \quad k^2 = k_1^2 + k_2^2 \tag{4}$$

On the boundaries of the layer we have

$$v = v' = 0 \qquad \text{for } z = \pm 1 \tag{5}$$

The boundary value problem (4) and (5) defines the spectrum of characteristic perturbations and of the corresponding decrements λ. It should be noted that (4) differs from the Orr-Sommerfeld equation which occurs in the investigation of perturbations of longitudinal flows.

We shall first show that all perturbations of the type (3) decay with time. To do this, we shall multiply (4) by the complex conjugate solution v and integrate the result with respect to z from -1 to 1. Combining the result with the complex conjugate, we obtain

$$(\lambda + \lambda^*) \int_{-1}^{1} (|v'|^2 + k^2 |v|^2)\, dz = 2 \int_{-1}^{1} (|v''|^2 + 2k^2 |v'|^2 + k^4 |v|^2)\, dz \tag{6}$$

Since both integrals entering (6) are positive, we have $\lambda + \lambda^* > 0$. Thus, real parts λ_r of the decrements of all normal perturbations are positive and the steady motion (1) is always stable.

To find the spectrum of decrements and perturbations, we must solve the boundary value problem (4) and (5) which is not self-conjugate.

General solution of the linear equation with constant coefficients (4) has the form

$$v = C_1 e^{kz} + C_2 e^{-kz} + C_3 e^{r_1 z} + C_4 e^{r_2 z} \tag{7}$$

$$r_{1,2} = {}^1/_2 [R \pm \sqrt{R^2 + 4(k^2 - \lambda)}] \tag{8}$$

Boundary conditions (5) result in the set of four linear homogeneous equations for the coefficients C_i. Condition of solvability of this system gives a characteristic equation which can be written as

$$\frac{\tanh^2 k - \tanh r_1 \tanh r_2}{\tanh r_1 - \tanh r_2} = \frac{(k^2 - r_1 r_2)\tanh k}{k(r_1 - r_2)} \tag{9}$$

Separating the real and imaginary part, we obtain

$$\frac{\alpha (k^2 + \alpha^2 + \beta^2 - {}^1/_4 R^2)}{4(\alpha^2 + \beta^2)} \frac{\sinh 2k}{k} + \frac{\sinh 2\alpha}{\cosh 4\alpha - \cos 4\beta} (\cosh R \cos 2\beta - \cosh 2k \cosh 2\alpha) = 0 \tag{10}$$

$$\frac{\beta (k^2 - \alpha^2 - \beta^2 - {}^1/_4 R^2)}{4(\alpha^2 + \beta^2)} \frac{\sinh 2k}{k} + \frac{\sin 2\beta}{\cosh 4\alpha - \cos 4\beta} (\cosh R \cosh 2\alpha - \cosh 2k \cos 2\beta) = 0 \tag{11}$$

Real magnitudes α and β entering (10) and (11) can be found from

$$ {}^1/_2 \sqrt{R^2 + 4(k^2 - \lambda)} = \alpha + i\beta \tag{12}$$

Assuming $\lambda = \lambda_r + i\lambda_i$ we have

$$\lambda_r = k^2 - \alpha^2 + \beta^2 + 1/4\, R^2, \qquad \lambda_i = -2\alpha\beta \tag{13}$$

Thus, Equations (10) and (11) define real and imaginary parts of characteristic decrements λ_r and λ_i in terms of R and k. Coefficients C_i of the solution of (7) defining the form of perturbations are of unwieldy appearance and are, consequently, omitted.

When $R = 0$, the decrements λ of normal perturbations are real and positive (monotonic decay of perturbations). They can be determined (in order of increasing magnitude) from the following transcendental relations :

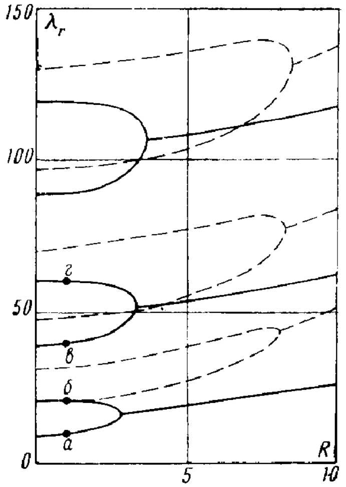

Fig. 1

$$\beta_0 \tan \beta_0 = -k \tanh k, \quad \lambda^{(0)} = \lambda_0^{(0)}, \quad \lambda_2^{(0)}, \ldots$$
$$\beta_0 \cot \beta_0 = k \coth k, \quad \lambda^{(0)} = \lambda_1^{(0)}, \ \lambda_3^{(0)}, \ldots$$
$$(\beta_0 = \sqrt{\lambda^{(0)} - k^2}) \tag{14}$$

Here, the solutions of (4) with $R = 0$ which are even with respect to the middle of the layer, correspond to the first subsystem, while odd solutions correspond to the second subsystem (see e. g. [1]).

At small values of R the decrements λ remain real, and in this region $\lambda_i = 0$, i. e. $\alpha = 0$. In this case (10) is satisfied identically and λ_r can be found from (11), which becomes

$$\left(k^2 - \beta^2 - \frac{R^2}{4}\right) \frac{\sinh 2k}{2k} + \tag{15}$$
$$+ \frac{\beta}{\sin 2\beta} (\cosh R - \cosh 2k \ \cos 2\beta) = 0$$

From (15) it follows that for small R, the decrements vary with increasing R, as R^2

$$\lambda = \lambda^{(0)} + aR^2 + \ldots \tag{16}$$
$$a = \beta_0 (4k\beta_0 - \sinh 2k \sin 2\beta_0) \, [\beta_0 \sin 2\beta_0 \sinh 2k \, (1 - 2 \, k \coth 2k) -$$
$$- k \, (1 - 2\beta_0 \cot 2\beta_0) \, (1 - \cosh 2k \cos 2\beta_0)]^{-1} + {}^1/_4 \tag{17}$$

With R increasing, the real decrements merge pairwise, forming complex conjugate pairs of decrements, i. e. oscillatory perturbations appear and move along the layer with phase velocity different from zero.

The spectrum under consideration is thus analogous to the spectrum of decrements in case of longitudinal flows with an odd profile [1].

Fig. 1 shows, as an example, spectra constructed according to (10) and (11) for wave numbers $k = 1$ (solid line) and $k = 4$ (dotted line).

In the region of large R, (10) and (11) can yield asymptotic relationships between α and β (i. e. λ_r and λ_i) and the Reynolds' number R. We find that for large R, parameter β is independent of R, while α is linearly dependent on R

$$\alpha = \frac{R}{2} - \delta, \qquad \delta = \ln \left(\frac{\beta \sinh 2k}{k \sin 2\beta}\right)^{1/2} \tag{18}$$

and the values of β are given as roots of the transcendental equation

$$2\beta \cot 2\beta + \ln \frac{\sin 2\beta}{2\beta} = 2k \coth 2k + \ln \frac{\sinh 2k}{2k} \tag{19}$$

Each root β_n of (19) corresponds to a complex conjugate pair formed as the result of merging of two neighboring real levels of the spectrum. For $k = 1$ for example, we have $\beta = 3.632$, 6.883, 10.07, 13.24... and corresponding $\delta = 1{,}382$, 1.644, 1.819, 1.948... For $k = 4$, we have $\beta = 3.338$, 6.624, 9.860, 13.07... and $\delta = 4.043$, 4.137, 4.239, 4.331... Real and imaginary parts of decrements in the region of large R can be found in accordance with (13), from

$$\lambda_r = (k^2 + \beta^2 - \delta^2) + \delta R, \qquad \lambda_i = \pm (R - 2\delta)\,\beta \tag{20}$$

It should be noted that the asymptotic formulas (20) describe well the whole complex region of the spectrum. In particular, the values of Reynolds' number R_* for which oscillatory perturbations occur (at this point $\lambda_i = 0$) can be found with sufficient accuracy from Formula $R_* = 2\delta$ which follows from (20).

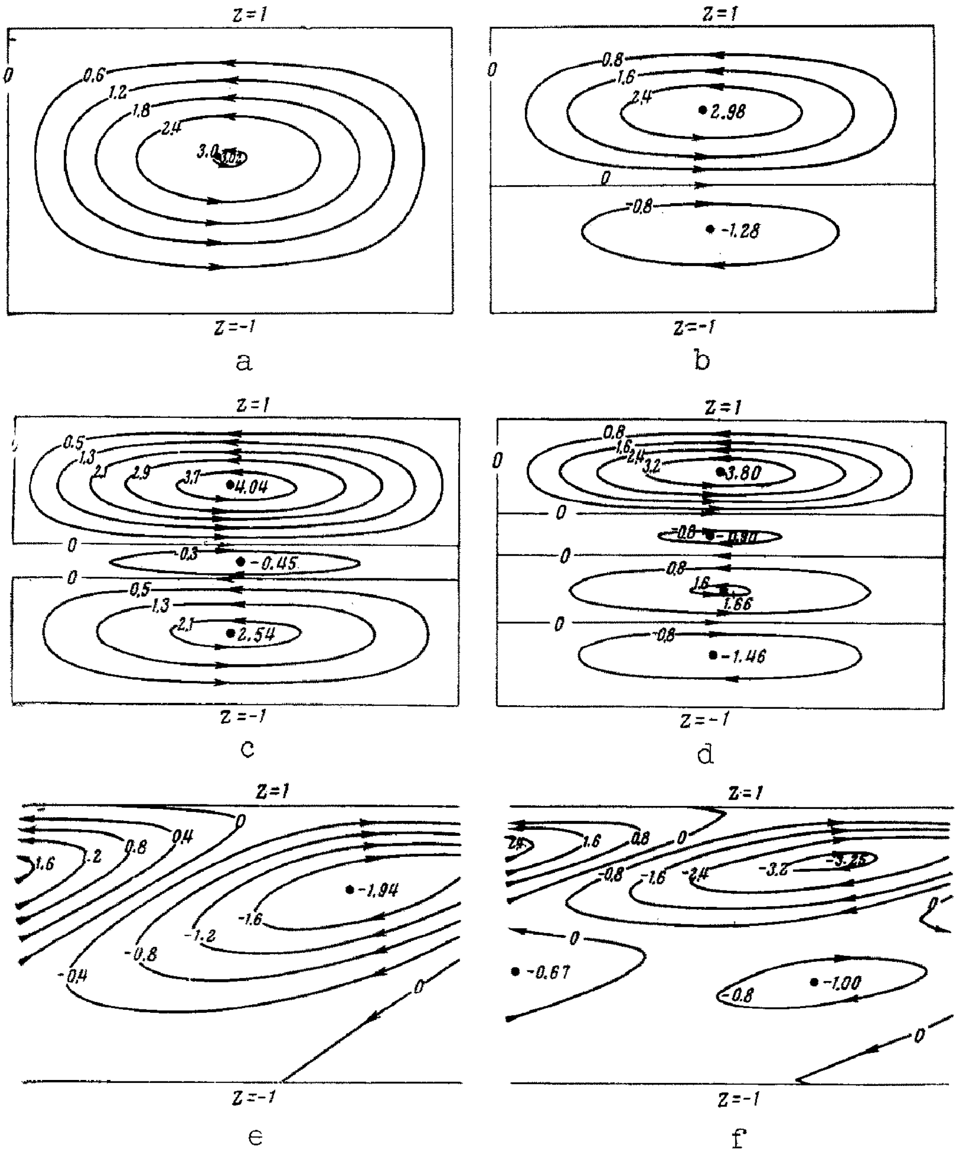

Fig. 2

Fig. 2 gives the perturbation streamlines in the plane case ($k_2 = 0$, $k = k_1 = 1$)

for a fixed moment of time. Graphs show the pattern of motion in the interval of values of x equal to half of the wavelength of the perturbation wave. Fig. 2a to d correspond to the decaying monotone perturbations for various levels of the spectrum when $R = 1$ (values of R and λ_r to which these figures correspond are denoted on Fig. 1 by the bold type dots). Fig. 2e and f show the streamlines of decaying oscillatory perturbations

$$\operatorname{Re} \psi(x, z) = \text{const}$$

(Here ψ is the complex stream function) for the first (Fig. 2e) and the second (Fig. 2f) in the direction of increasing λ_r, pair of merged levels for $R = 19$.

With the increasing number of levels the structure of eigenfunctions becomes more complex, and the effect of sidewise displacement of perturbations due to the transverse flow becomes noticeable.

BIBLIOGRAPHY

1. Birikh, R. V., Gershuni, G. Z. and Zhukhovitskii, E. M., O spektre vozmushchenii ploskoparallel'nykh techenii pri malykh chislakh Reinol'dsa (On the spectrum of perturbations of plane-parallel flows at low Reynolds' numbers). PMM Vol. 29, No. 1, 1965.

Translated by L. K.